数理解析研究所講究録 1271 巻 2002 年 32-40

高調波共鳴定在波の線形安定性

九大応力研 (RIAM, Kyushu Univ.) 岡村 誠 (OKAMURA Makoto)
IRD Mansour IOUALALEN
IRPHE Christian KHARIF

Stability calculations of three-dimensional short-crested waves very near their two-dimensional standing wave limit are performed on water of uniform depth. Non-resonant waves are stable while resonant waves are unstable, which means that the resonant interaction contributes to instability.

1 Introduction

Linearly, a short-crested wave is defined as superposition between an incident travelling wave with an angle θ to a vertical wall and its reflected wave, where θ is the angle between the direction of the incident wave and the normal to the wall. The standing wave limit corresponds to the angle $\theta=0°$.

The properties of short-crested waves have been discussed in Marchant & Roberts (1987) on water of finite depth. The authors conjectured that short-crested wave fields may be unstable through harmonic resonance phenomena. Later, Ioualalen *et al.* (1996) showed that harmonic resonance is associated with sporadic and weak superharmonic instability for short-crested waves in finite depth. In particular the instability region exhibits a bubble-like shape in the wave steepness parameter space. However their short-crested wave solutions were not complete because only one branch of the multiple-like solutions associated with harmonic resonance has been analysed. Ioualalen & Okamura (2002) calculated nonlinear short-crested waves with multiple-like solutions, *i.e.*, two branches linked by a turning point and one single branch. They found the solutions by Ioualalen *et al.* (1996) incomplete when harmonic resonance occurs. They also obtained their stability diagram in the vicinity of harmonic resonance and found that harmonic resonance is associated with two bubbles of instability that are not anymore sporadic.

In the present study, we examine the relation between short-crested waves with a small angle θ and standing waves near the critical depth. Then we perform a superharmonic stability analysis of resonant short-crested waves very near the standing wave limit. Our stability scheme does not apply directly to standing waves in order to use the stability analysis for steady waves.

2 Formulation

We consider standing gravity waves on an inviscid, incompressible fluid of finite depth where the flow is assumed irrotational. The governing equations are given in a dimensionless form with respect to the reference length $1/k$ and the reference time $(gk)^{-1/2}$, where g is the gravitational acceleration and k the wavenumber of the incident wave train.

Let us define a frame of reference (x^*, y^*, z^*, t^*, ϕ^*) so that $x^* = x - ct$, $y^* = y$, $z^* = z$, $t^* = t$ and $\phi^* = \phi - cx^*$, where c represents the propagation velocity of the short-crested wave train and is equal to ω/α, ω being the frequency of the wave and $\alpha = \sin\theta$ is the x-direction wave number, the y-direction wave number being $\beta = \cos\theta$. If we omit the asterisks for sake of simplicity, the governing equations are:

$$\Delta\phi = 0, \quad \text{for } -d < z < \eta, \tag{1}$$

$$\phi_z = 0, \quad \text{on } z = -d, \tag{2}$$

$$\phi_t + \eta + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2 - c^2) = 0, \quad \text{on } z = \eta, \tag{3}$$

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0, \quad \text{on } z = \eta, \tag{4}$$

where d is the depth of the fluid, $\phi(x, y, z, t)$ the velocity potential and $z = \eta(x, y, t)$ the equation of the free surface.

We introduce the following functions to construct a stability problem:

$$\eta(x, y, t) = \bar{\eta}(x, y) + \eta'(x, y, t), \tag{5}$$

$$\phi(x, y, z, t) = \bar{\phi}(x, y, z) + \phi'(x, y, z, t), \tag{6}$$

where we assume that the surface elevation and the velocity potential are superposition of a steady unperturbed wave $(\bar{\eta}, \bar{\phi})$ and infinitesimal perturbations (η', ϕ') where $\eta' \ll \bar{\eta}$ and $\phi' \ll \bar{\phi}$. After substituting expressions (5) and (6) into equations (1)–(4) and linearizing, we obtain the zeroth order system of equations for which permanent short-crested waves are solutions and the first order perturbation equations representing the stability problem.

In order to solve the zeroth order system of equations, we look for the following form of the velocity potential:

$$\bar{\phi} = -cx + \sum_{k=0}^{N} \sum_{j=2-(k \bmod 2)}^{N} \phi_{jk} \sin(j\alpha x)\cos(k\beta y)\frac{\cosh[\kappa_{jk}(z+d)]}{\cosh(\kappa_{jk}d)}, \tag{7}$$

where $\kappa_{JK} = [(J\alpha)^2 + (K\beta)^2]^{1/2}$ and N is the maximum order of expansion and is chosen to be 19 in this paper. Further details about the computations of the short-crested waves can be found in Okamura (1996).

The first order system of equations is

$$\Delta\phi' = 0, \quad \text{for } -d < z < \bar{\eta}, \tag{8}$$

$$\phi'_z = 0, \quad \text{on } z = -d, \tag{9}$$

$$\phi'_t = -\bar{\phi}_x\phi'_x - \bar{\phi}_y\phi'_y - \bar{\phi}_z\phi'_z - \eta'(1+\bar{\phi}_x\bar{\phi}_{xz}+\bar{\phi}_y\bar{\phi}_{yz}+\bar{\phi}_z\bar{\phi}_{zz}), \quad \text{on } z = \bar{\eta}, \tag{10}$$

$$\eta'_t = \eta'(\bar{\phi}_{zz}-\bar{\eta}_x\bar{\phi}_{xz}-\bar{\eta}_y\bar{\phi}_{yz})-\bar{\eta}_x\phi'_x-\bar{\phi}_x\eta'_x-\bar{\eta}_y\phi'_y-\bar{\phi}_y\eta'_y+\phi'_z, \quad \text{on } z = \bar{\eta}. \tag{11}$$

We look for non-trivial solutions of the following form:

$$\eta' = e^{-i\sigma t} \sum_{J=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} a_{JK} e^{i(J\alpha x + K\beta y)}, \tag{12}$$

$$\phi' = e^{-i\sigma t} \sum_{J=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} b_{JK} e^{i(J\alpha x + K\beta y)} \frac{\cosh[\kappa_{JK}(z+d)]}{\cosh(\kappa_{JK} d)}, \tag{13}$$

which is reduced to the eigenvalue problem determining the eigenvalues σ and their eigenvectors consisting of a_{JK} and b_{JK}.

3 Relation between standing and short-crested waves

Marchant & Roberts (1987) showed that harmonic resonance occurs for standing waves of finite depth when a harmonic (m, n) is a solution of the homogeneous differential equation derived from the surface conditions. Such case occurs at critical depths d which satisfy the relation,

$$n \tanh(nd) = m^2 \tanh d. \tag{14}$$

The lowest order harmonic resonance occurs at depth $d_{\rm hr} \approx 0.624$ which is related to harmonic resonance $(3, 5)$.

We analyse the $(3, 5)$ resonance because it is the strongest harmonic resonance. Figure 1 exhibits the multiple-like solution structure of the coefficient ϕ_{35} as a function of the coefficient ϕ_{11} of the fundamental mode

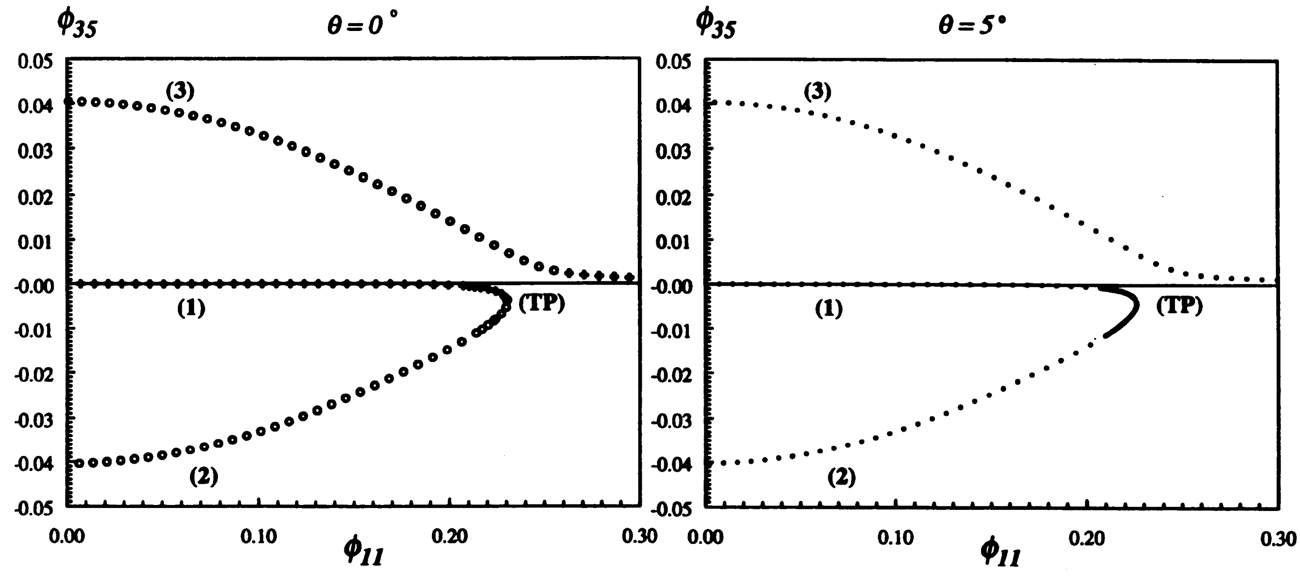

Figure 1: Coefficient ϕ_{35} versus coefficient ϕ_{11} for depth $d = 0.58$ and angles $\theta = 0°$ (left) and $\theta = 5°$ (right). Circle-signs ($\circ$) and plus-signs ($+$) denote the unstable and stable solutions, respectively (displayed only for $\theta = 0°$).

for depth $d = 0.58$ at angles $\theta = 0°$ and $\theta = 5°$. The solutions are composed of three branches: branches (1) and (2) linked by a turning point (TP) and branch (3). The figure shows that the solutions for $\theta = 0°$ are very similar to those for $\theta = 5°$ and thus we can use the short-crested waves for $\theta = 0.001°$ to obtain the results for the stability of standing waves.

Figure 1 also indicates that the resonant harmonic mode ϕ_{35} is relatively dominant both on branch (2) and on branch (3) for ϕ_{11} smaller than the turning point (TP). We call it resonant wave. However the fundamental mode ϕ_{11} is relatively dominant both on branch (1) and on branch (3) for ϕ_{11} larger than the turning point (TP). We call it non-resonant wave.

4 Superharmonic instability of short-crested waves near their standing wave limit: $\theta = 0.001°$

We perform here the superharmonic instabilities of short-crested waves that are very close to standing waves; that is, angle $\theta = 0.001°$. The aim of this study is to characterize the superharmonic instability associated with harmonic resonance appearing in standing waves as Ioualalen & Okamura (2002) clarified the relation between the superharmonic instability and harmonic resonance for short-crested waves. The time scale of the strongest instability tells us whether the multiple-like solution related to harmonic resonance is observable or not.

A superharmonic instability associated with a harmonic resonance (m, n) can arise only if the two eigenvalues with opposite signature are equal,

$$\sigma^{s}_{m,n}(h) = \sigma^{-s}_{-m,n}(h), \tag{15}$$

for some wave steepness h. For standing waves the condition of harmonic resonance is equivalent to condition (14). Such superharmonic instability is described as an interaction between the two eigenmodes $(\pm m, n)$ and the $2m$-modes $(1, \pm 1)$ of the basic unperturbed standing wave, that is,

$$\Omega_1 = -\Omega_2 + m\Omega_{01} + m\Omega_{02}, \tag{16}$$

$$\boldsymbol{k}_1 = \boldsymbol{k}_2 + m\boldsymbol{k}_{01} + m\boldsymbol{k}_{02}, \tag{17}$$

where $\Omega_i = [|\boldsymbol{k}_i| \tanh(\kappa_{mn} d)]^{1/2}$, $\Omega_{0i} = \tanh^{1/2} d$ for $i = 1, 2$ and $\boldsymbol{k}_1 = (\alpha m, \beta n)$, $\boldsymbol{k}_2 = (-\alpha m, \beta n)$, $\boldsymbol{k}_{01} = (\alpha, \beta)$, and $\boldsymbol{k}_{02} = (\alpha, -\beta)$.

In Figures 2 and 3 are plotted the frequencies and growth rates of the eigenvalues $\sigma_{\pm 3,5}$ for all branches of the wave solutions for depths $d = 0.58$ and $d = 0.62$ in the vicinity of the critical depth $d_{\mathrm{hr}} \approx 0.624$. For both

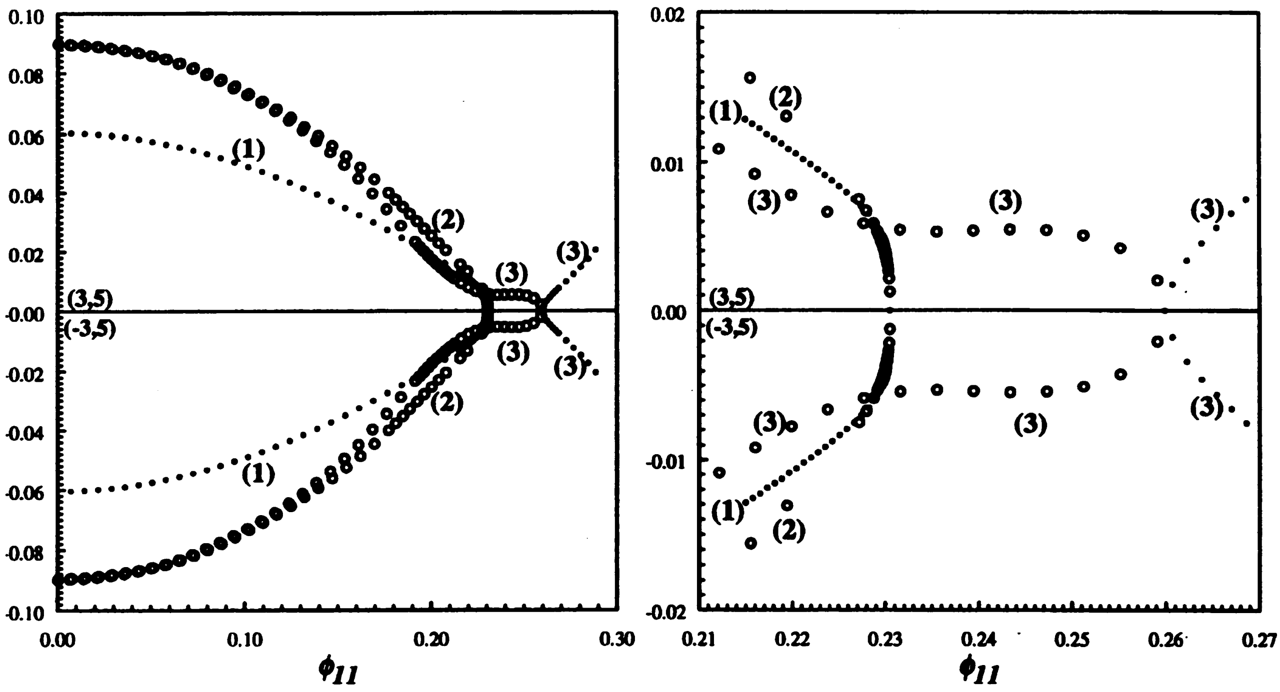

Figure 2: Frequency $[-\Re(\sigma_{\pm35})]$ ($\bullet$) and growth rate $[-\Im(\sigma_{\pm35})]$ ($\circ$) as a function of coefficient ϕ_{11} for angle $\theta = 0.001°$ and depth $d = 0.58$. The right panel is an enlargement of the left panel.

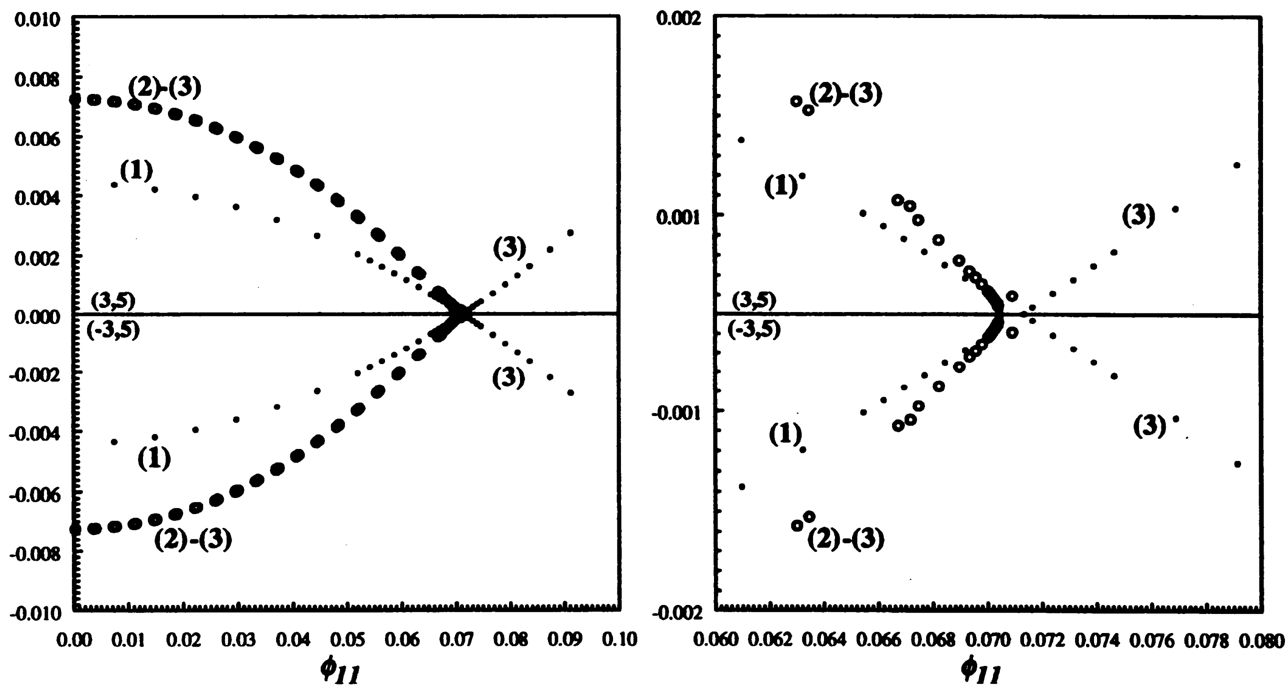

Figure 3: The same as Figure 2 except for depth $d = 0.62$.

depths, branch (1) is stable on its whole region, from $\phi_{11} = 0$ to the turning point ($\phi_{11} \approx 0.2305605$ for $d = 0.58$ and $\phi_{11} \approx 0.0705$ for $d = 0.62$), while branch (2) is unstable on its whole region. The transition from stable to unstable occurs when the frequency reaches the zero-axis, then the growth rate value leaves it. For both depths the dominant instability appears for $\phi_{11} = 0$ and the instability on branch (2) weakens with increasing ϕ_{11} to disappear at the turning point (here at the zero-axis). Branch (3) is unstable from $\phi_{11} = 0$ to the turning point ahead ($\phi_{11} \approx 0.2591$ for $d = 0.58$ and $\phi_{11} \approx 0.0709$ for $d = 0.62$). The maximum of instability also appears for $\phi_{11} = 0$. The instability occurs when eigenvalues $\sigma_{3,5}$ and $\sigma_{-3,5}$ coalesce at zero-frequency (phase-locked with the unperturbed wave). Such instability is physically associated with a resonant interaction: the coalescence of the two eigenmodes at zero-frequency simply means that the harmonics $(\pm 3, 5)$ propagate at the same phase speed as the basic wave, bearing in mind that the stability problem has been computed in the frame of reference moving with the basic wave.

Ioualalen & Okamura (2002) showed that for resonant short-crested waves the instability region is a small range of ϕ like a bubble. In the present case the instability region is a wide range of ϕ_{11}, which is much different from that in the short-crested waves. The instability is strong for resonant wave, *i.e.*, on branch (2) and the left part of branch (3). The instability weakens as ϕ_{11} becomes larger. Beyond the turning point the solution on branch (3) remains weakly unstable within a certain range of the parameter regime then it turns stable.

5 Conclusion

This study deals with the stability of the two-dimensional standing waves with multiple-like solutions for the strongest harmonic resonance (3,5) occurs. Since our numerical procedure calculating the stability of three-dimensional short-crested waves does not apply to two-dimensional standing waves because the waves are not anymore stationary, we first show that short-crested waves and standing waves match each other at the limit ($\theta \to 0°$) in order to extend the stability results here to standing waves. Then we perform a superharmonic stability analysis of short-crested waves very near their standing wave limit. The stability analysis shows that resonant waves are strongly unstable. By contrast, non-resonant waves are almost stable and weakly unstable within a sporadic range of the parameter region then non-resonant waves are therefore only solutions to exist.

参考文献

[1] M. Ioualalen & M. Okamura, "Structure of the instability associated with harmonic resonance of short-crested waves." *J. Phys. Oceanogr.* **32** (2002) 1331–1337.

[2] M. Ioualalen, A. J. Roberts & C. Kharif, "On the observability of finite depth short-crested water waves." *J. Fluid Mech.* **322** (1996) 1–19.

[3] T. R. Marchant & A. J. Roberts, "Properties of short-crested waves in water of finite depth." *J. Austral. Math. Soc.* **B 29** (1987) 103–125.

[4] M. Okamura, "Notes on short-crested waves in deep water." *J. Phys. Soc. Jpn.* **65** (1996) 2841–2845.